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Periodic Boundary Value Problems On Thermoelastic Star Graphs And Their Solutions By Use General Function Method

Nursaule Zhumatkyzy Ainakeyeva*, Dadayeva A.N†

Joldasbekov Institute of Mechanics and Engineering, Almaty, Kazakhstan

Abstract: This paper presents a comprehensive study on periodic boundary value problems (BVPs) in thermoelastic star graphs, utilizing the method of generalized functions. The research focuses on the behavior of rod structures subjected to thermal heating and cooling, providing a unified approach to solving various boundary value problems relevant to practical applications. We derive integral representations of generalized solutions that facilitate the determination of displacements, deformations, stress, temperature, and heat fluxes across each element of the graph. The study also addresses the modeling of force and heat sources through both regular and singular generalized functions under diverse boundary conditions. By establishing continuity and Kirchhoff conditions at the common node of the graph, we formulate the governing equations for the amplitudes of displacement and temperature. The findings highlight the versatility of the proposed method, which can be extended to a wide range of network structures, distinguishing it from existing techniques. This work contributes to the understanding of thermoelastic behavior in complex systems, with implications for engineering applications in fields such as mechanical design, materials science, and structural analysis.

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1. Introduction

Graph theory has wide applications in subjects such as economics, logistics, sociology, optimal control, and navigation [1-2]. The properties of graphs are also actively used to solve boundary value problems (BVPs) on network-like structures, e.g., oil pipelines, gas pipelines, and electrical networks [3-10]. With the development of mechanical engineering, complex multi-link rod structures operating under various thermal conditions began to be actively used. They are widely used in structural mechanics, mechanical engineering, robotics, and many other fields. An urgent scientific and technical task is to study the thermally stressed state of network systems for various purposes under dynamic and thermal influences, taking into account their thermoelastic properties under dynamic and thermal influences, including impact types. This is necessary to analyze the strength and reliability of such objects, determine safe operating modes, and prevent disasters. Mathematical modeling of the thermodynamics of rod structures and the creation of information technologies based on it is one of the more effective and inexpensive methods for researching and designing such systems. Here boundary value problems of uncoupled thermoelasticity are considered on a star thermoelastic graph (Fig. 1), which can be used to study various mesh structures under conditions of volume and thermal heating (cooling).

*Joldasbekov Institute of Mechanics and Engineering, Almaty, Kazakhstan. **Corresponding Author: alexeeva@math.kz**
† Institute of Mathematics and Mathematical Modelling, Pushkin Street, 125 28, Almaty 050010, Kazakhstan. **Email: 5dady1262@mail.ru**.
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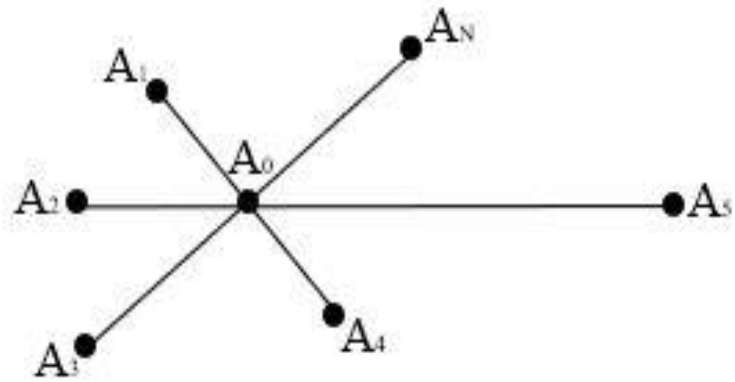

Figure-1 Star graph

The novelty of the present work lies in the fact that a generalized function method is used to solve boundary value problems, leading to a differential equation solution with a singular right-hand side **[11]**. The solution is constructed as the convolution of the Green's function of the equation with the appropriate right-hand side. To determine the unknown boundary values of the solution and its derivatives on each segment, resolving boundary equations are constructed at the ends, employing the asymptotic properties of Green's function and its derivative at zero. To construct a closed system of equations, the obtained algebraic equations for each edge of the graph are supplemented with transmission conditions at the node and linear boundary conditions at its ends. These conditions can be either locally or not locally connected. Thus, the proposed method applies to a wide range of BVPs, including those on mesh structures.

2. Statement of a boundary value problem on a thermoelastic star graph

We consider the periodic vibration of a thermo-elastic star graph with frequency ω. This graph contains N edges $S_j = (A_0, A_j)$ of the length $L_j \ (j = 1, 2, \ldots, N)$ with a common node A_0, $0 \le x \le L_j$. Amplitudes of displacement $u_j(x,t) = u_j(x)e^{-i\omega t}$ and temperature $\theta_j(x,t) = \theta_j(x)e^{-i\omega t}$ satisfy the equations **[12]**:

$$\begin{gathered}\frac{\partial^2 u_j}{\partial x^2} + \frac{\omega^2}{c_j^2} u_j - \tilde{\gamma}_j \frac{\partial \theta_j}{\partial x} + F_1^j(x) = 0, \\ \kappa_j \frac{\partial^2 \theta_j}{\partial x^2} + i\omega\theta_j - F_2^j(x) = 0\end{gathered} \tag{1}$$

Here γ_j, κ_j are the thermoelastic constants, $\tilde{\gamma}_j = \dfrac{\gamma_j}{\rho_j c_j^2}$, c_j is the velocity of elastic waves, ρ_j is the density of mass, $F_1^j(x,t) = F_1^j(x)e^{-i\omega t}$ is the longitudinal component of acting periodic force in a j-th edge of the graph; $F_2^j(x,t) = F_2^j(x)e^{-i\omega t}$ describes the power of heat sources on it.

The thermoelastic stress in the rod is determined by Duhamel-Neumann law:

$$\sigma_j(x,t) = \rho_j c_j^2 u_{j,x}(x,t) - \gamma_j \theta_j(x,t) \tag{2}$$

where $u_{j,x}(x,t) \equiv \frac{\partial u_j}{\partial x}$. Here we pose the following boundary value problem (BVP). Amplitudes of displacements and temperature are known at the ends of the graph: for all $j = 1,\ldots,N$

$$u_j\left(L_j,t\right) = w_2^j(\omega)e^{-i\omega t}, \quad \theta_j(L_j,t) = \theta_2^j(\omega)e^{-i\omega t} . \tag{3}$$

We enumerate by index 1 the point *x=0* and by index 2 the point *x=* L_j $\left(x_1 = 0,\ x_2 = L_j\right)$ and boundary displacements and temperature $\left(w_1^j\theta_1^j, w_2^j, \theta_2^j\right)$ at the ends of segments.

The following continuity conditions and generalized Kirchhoff conditions are specified in the common node A_0 of the graph:

$$w_1^1 = w_1^2 = \ldots = w_1^N, \qquad \theta_1^1 = \theta_1^2 = \ldots = \theta_1^N, \tag{4}$$

$$\sum_{j=1}^{N} E_j p_1^j = P(\omega), \quad \sum_{j=1}^{N} \kappa_j q_1^j = G(\omega). \tag{5}$$

$$p_1^j(\omega) = \left.\frac{\partial u_j}{\partial x}\right|_{x=0}, \quad p_2^j(\omega) = \left.\frac{\partial u_j}{\partial x}\right|_{x=L_j}$$

$$q_1^j(\omega) = \left.\frac{\partial \theta_j}{\partial x}\right|_{x=0}, \quad q_2^j(\omega) = \left.\frac{\partial \theta_j}{\partial x}\right|_{x=L_j}$$

3. Presentation of amplitudes of temperature and displacement at edges of a star graph in the space

To solve this problem we use the solutions of BVPs for the heat equation and wave equation on the segment , which were constructed by use the General Function Method in **[13-14]**.

The heat equation in (1) has the next form for $\hat{\theta}(x)$ in the space of general slow growth functions $S'(R^2)$:

$$-\kappa_j \frac{\partial^2 \hat{\theta}_j}{\partial x^2} - i\omega\hat{\theta}_j = \hat{F}_2^j(x) + \kappa_j q_2^j(\omega)\delta(L_j - x)H(x) - \kappa_j q_1^j(\omega)H(L_j - x)\delta(x) -$$
$$-\kappa_j \theta_2^j((\omega)\delta'(L_j - x)H(x) - \kappa_j \theta_1^j(\omega)\delta'(x)H(L_j - x) \tag{6}$$

Note that the right side of this equation includes boundary amplitude of temperature $\theta_j(\omega)$ and heat flows $\Pi_k^j(\omega) = \kappa_j q_k^j(\omega)$ (*k*=1, 2). Using the property of fundamental solution $U_2^j(x,\omega)$ of this equation we obtain its general solution as convolution the right part of (6) with $U_2^j(x,\omega)$ on every edge of graph (*j*=1…, *N*):

$$\theta_j(x,\omega) = F_2^j(x,\omega) \underset{x}{*} U_2^j(x,\omega) + \kappa_j q_2^j(\omega)H(x)U_2^j(L_j - x,\omega) -$$
$$-\kappa_j q_1^j(\omega)H(L_j - x)U_2^j(x,\omega) - \kappa_j\theta_2(\omega)H(x)U_{2,x}^j(L_j - x,\omega) - \kappa_j\theta_1(\omega)H(L_j - x)U_{2,x}^j(x,\omega); \tag{7}$$

$$\bar{U}_2^j(x,\omega) = -\frac{\sin k_j\,|x|}{2k_j}, \quad k_j = (1+i)\sqrt{\frac{\omega}{2\kappa_j}}.$$

The wave equation for displacement in (1) has the next form for $\hat{u}(x)$ in $S'(R^2)$:

$$c_j^2 \hat{u}^j,_{xx} + \omega \hat{u}^j = \tilde{\gamma}_j \hat{\theta}^j,_x - \hat{F}_1^j(\omega) + c_j^2 \left\{ w_1^j(\omega)\delta'(x) - w_2^j(\omega)\delta'(x-L_j) \right\} +$$
$$+ c_j^2 \left\{ \left(p_1^j(\omega) - \tilde{\gamma}\theta_1^j(\omega) \right)\delta(x) - \left(p_2^j(\omega) - \tilde{\gamma}\theta_2^j(\omega) \right)\delta(x-L_j) \right\} \quad (8)$$

Note that the right side of this equation includes boundary amplitude of displacements $w_k^j(\omega)$ and their derivatives $p_k^j(\omega)$ (k=1, 2). Using the property of fundamental solution $U_1^j(x,\omega)$ of this equation we obtain its general solution as convolution the right part of (8) with $U_1^j(x,\omega)$ on every edge of graph (j=1…, N):

$$u_j(x,\omega) H(L_j - x) H(x) = \mathrm{P}^j(x,\omega) - c_j^2 p_1^j(\omega) H(L_j - x) U_1(x,\omega) -$$
$$- c_j^2 w_1^j(\omega) H(L_j - x) U_1,_x(x,\omega) + c_j^2 p_2^j(\omega) H(x) U_1(L_j - x,\omega) + \quad (9)$$
$$+ c_j^2 w_2(\omega) H(x) U_1,_x(L_j - x,\omega)$$

where,

$$U_1^j(x,\omega) = -\frac{1}{2\omega c_j}\sin\left(\lambda_j |x|\right), \quad \lambda_j(\omega) = \frac{\omega}{c_j}; \quad (10)$$

4. Solution of BVP on the heat star graphs

If the boundary functions are known: $\mathbf{B}1^j(\omega) = \left(w_1^j, p_1^j, w_2^j, p_2^j\right)$, $\mathbf{B}2^j(\omega) = \left(\theta_1^j, q_1^j, \theta_2^j, q_2^j\right)$, then formulas (7) and (9) determine the temperature and displacement on each edge of the graph. From equations (7) and (9), by limiting to the ends of the edges, boundary equations are obtained that connect the boundary values of displacements, temperature, and their derivatives **[13-14]**.

We have the equations of connection boundary temperature at S_j edges and their derivatives of the thermal graph **[12]**:

$$\mathbf{A}2^j(\omega) \times \mathbf{B}2^j(\omega) = \mathbf{F}2^j(\omega), \quad (11)$$

$$\mathbf{A}2^j(\omega) = \begin{bmatrix} 1 & 0 & -\cos(k_j L_j) & \dfrac{\sin(k_j L_j)}{k_j(\omega)} \\ -\cos(k_j L_j) & -\dfrac{\sin(k_j L_j)}{k_j(\omega)} & 1 & 0 \end{bmatrix},$$

$$\mathbf{F}2^j(\omega) = 2\left(F_2^j \underset{x}{*} U_2^j \Big|_{x=0}, \quad F_2^j \underset{x}{*} U_2^j \Big|_{x=L_j} \right).$$

The resolving system of equations for the Dirichlet boundary value problem on a thermal star graph with N edges has the form:

$$\mathbf{A}2(\omega) \cdot \mathbf{B}2(\omega) = \mathbf{F}(\omega), \quad (12)$$

where

$$\mathbf{B}2(\omega) = \left(\mathbf{B}2_1, \mathbf{B}2_2, ..., \mathbf{B}2_N\right), \quad \mathbf{F}2(\omega) = \left\{ \mathbf{F}2^1, ..., \mathbf{F}2^N; T^1, ..., T^N; \underbrace{0, ..., 0}_{N-1}; G(\omega) \right\}, \quad T^j = \theta_j(L_j)$$

Here the matrices have the following dimensions: $[\mathbf{A}2]_{4\,\mathrm{N}\times 4\,\mathrm{N}}, [\mathbf{B2}(\omega)]_{4\,\mathrm{N}\times 1}, [\mathbf{F}2(\omega)]_{4\,\mathrm{N}\times 1}$. The first 2N lines along the diagonal $\mathbf{A2}$ contain connection matrices for each edge of this graph $\mathbf{A}2_j$. The remaining elements are zero.

The next N-1 rows of the matrix $\mathbf{A2}$ contain the continuity conditions $(4)_2$. The last row of the matrix contains the Kirchhoff condition $(5)_2$ at the node A_0 of the star graph. The solution to algebraic equations (10) has the form:

$$\mathbf{B}2(\omega) = \mathbf{A}2^{-1} \times \mathbf{F}2(\omega)\,. \tag{13}$$

After determining the unknown edge and nodal functions $\mathbf{B}2(\omega)$, using formulas (6) we determine the temperature on any edge of the graph. (j=1..., N).

The boundary value problem on the thermal graph has been solved.

11. Solution of BVP on the elastic star graphs

There are next equations of connection boundary temperature at S_j edges of elastic graph [13, 14]:

$$\mathbf{A}1^j \times \mathbf{B}1^j = \mathbf{C}1^j(\omega), \tag{14}$$

$$\mathbf{A}1^j(\omega) = \begin{bmatrix} 1 & 0 & -\cos(\lambda_j L_j) & \dfrac{\sin(\lambda_j L_j)}{\lambda_j(\omega)} \\ -\cos(\lambda_j L_j) & \dfrac{-\sin(\lambda_j L_j)}{\lambda_j(\omega)} & 1 & 0 \end{bmatrix},$$

$$\mathbf{F}1^j(\omega) = -2\left(\left(F_1^j - \tilde{\gamma}_j \partial_x \theta_j\right) \underset{x}{*} U_1^j\Big|_{x=0},\ \left(F_1^j - \tilde{\gamma}_j \partial_x \theta_j\right) \underset{x}{*} U_1^j\Big|_{x=L_j}\right), \lambda_j(\omega) = \omega / c_j.$$

The resolving system of equations for the Dirichlet boundary value problem on an elastic star graph with N edges has the form:

$$\mathbf{A}1(\omega) \cdot \mathbf{B}1(\omega) = \mathbf{F}(\omega)\,, \tag{15}$$

Where,

$$\mathbf{B}1(\omega) = (\mathbf{B}1_1, \mathbf{B}1_2, ..., \mathbf{B}1_N),$$

$$\mathbf{F}1(\omega) = \left\{\mathbf{F}1^1, ..., \mathbf{F}1^N; d^1, ..., d^N; \underbrace{0, ..., 0}_{N-1}; P(\omega)\right\}.$$

$$d^1 = u_j\left(L_j\right)$$

Here the matrices have the following dimensions: $[\mathbf{A1}]_{4\,\mathrm{N}\times 4\,\mathrm{N}}, [\mathbf{B}1(\omega)]_{4\,\mathrm{N}\times 1}, [\mathbf{F}1(\omega)]_{4\,\mathrm{N}\times 1}$. The first $2N$ lines along the diagonal $\mathbf{A}1$ contain the connection matrices $\mathbf{A}1_j$ for each edge of this graph. The remaining elements are zero. The next *N-1* rows of the matrix $\mathbf{A1}$ contain the continuity conditions $(4)_2$. The last row of the matrix contains the Kirchhoff condition $(5)_2$ at the node A_0.
The solution to algebraic equations (15) has the form:

$$\mathbf{B}1(\omega) = \mathbf{A}1^{-1} \times \mathbf{F}1(\omega)\,. \tag{16}$$

After determining the unknown boundary and nodal functions $\mathbf{B}1(\omega)$, using formulas (7), we determine the temperature on any edge of the graph.

The boundary value problem on the elastic graph has been solved.

12. Conclusion

The method of generalized functions is used to solve boundary value problems of thermoelasticity on a star graph, which can be used to study various rod structures under conditions of thermal heating (cooling). A unified technique for solving various boundary value problems typical for practical applications has been developed.The action of force sources and heat sources can be modeled by both regular and singular generalized functions under various boundary conditions at the ends of the graph edge. The obtained regular integral representations of generalized solutions allow one to determine displacements, deformations, stress, temperature, and heat fluxes on each element of the graph, at any point. The method of generalized functions presented here allows one to solve a wide class of boundary value problems with local and related boundary conditions at the ends of the graph edges and various transfer conditions in its node and can be extended to network structures of various types. This distinguishes this method from all others that are used to solve similar problems.

For example, it is easy to construct the solutions of BVP on the graphs like type:

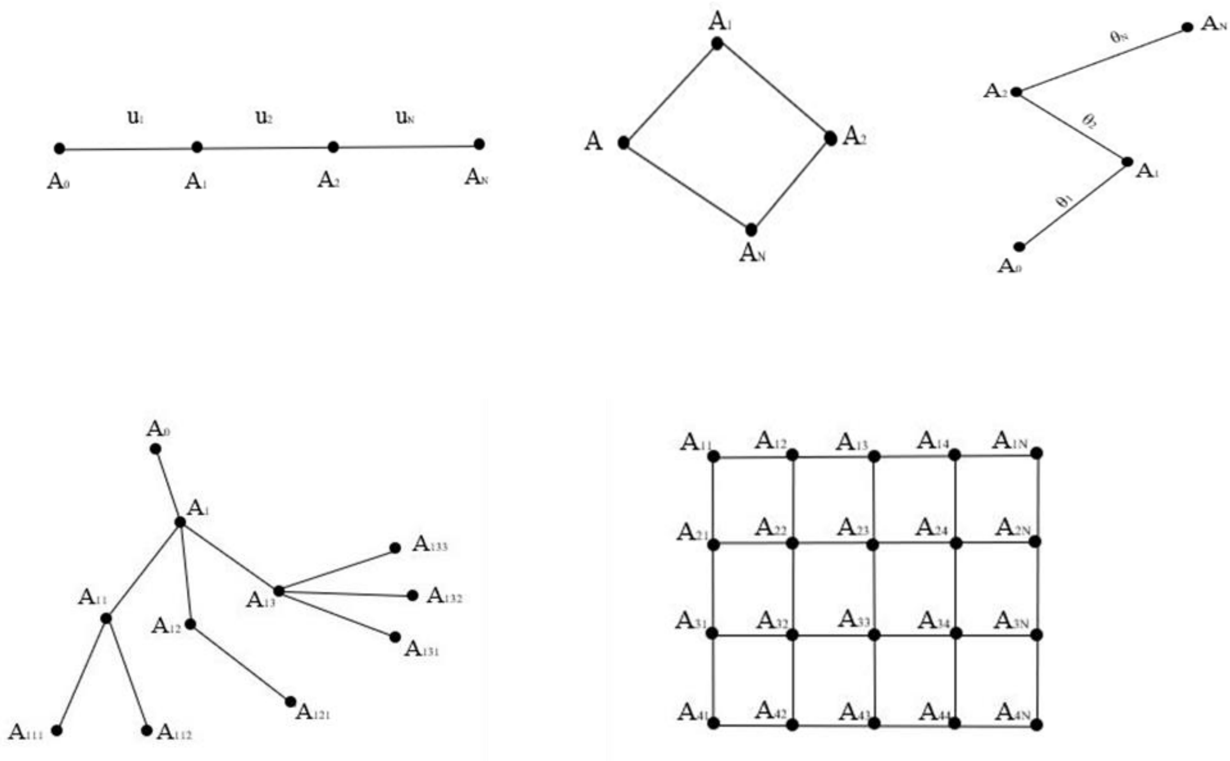

Figure-2 Examples of Linear, Star, Network and Lattice Graphs

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14. Conflict of Interest

The author declares no competing conflict of interest.

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